Categorified Crystal Operators on $U(\widehat{\mathfrak{sl}_p})$

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1 Preliminaries

Let $d\mathcal{H}_n$ be the degenerate affine hecke algebra and $L(a^n) = \operatorname{Ind}_{P_n}^{d\mathcal{H}_n} L(a) \boxtimes \ldots \boxtimes L(a)$. Recall that $L(a^n)$ is irreducible.

Definition 1.1. *Given $M \in d\mathcal{H}_n - mod$ and $a \in \Bbbk$ let $\Delta_a(M) =$generalized $a-$eigenspace for x_n on M, aka*

$$\Delta_a(M) := \bigoplus_{\underline{a} \in \Bbbk^n,\ a_n = a} M_{\underline{a}}$$

Lemma 1.2. *$\Delta_a : d\mathcal{H}_n - mod \to d\mathcal{H}_{n-1,1} - mod$ is an exact functor.*

Proof. Because x_n commutes with $d\mathcal{H}_{n-1,1}$, it first follows that $\Delta_a(M)$ will be a $d\mathcal{H}_{n-1,1}$ module and second, any $d\mathcal{H}_{n-1,1}$ morphism $M \to N$ restricts to $\Delta_a(M) \to \Delta_a(N)$. ∎

Definition 1.3. *More generally, define $\Delta_{a^m} : d\mathcal{H}_n - mod \to d\mathcal{H}_{n-m,m} - mod$ to be the simultaneous generalized $a-$aigenspace of $\{x_k\}_{k=n-m+1}^n$, aka*

$$\Delta_{a^m}(M) := \bigoplus_{\underline{a} \in \Bbbk^n,\ a_{n-m+1} = \ldots = a_n = a} M_{\underline{a}}$$

Lemma 1.4.

$$\operatorname{Hom}_{d\mathcal{H}_n}(\operatorname{Ind}_{n-m,m}^n (N \boxtimes L(a^m)), M) \cong \operatorname{Hom}_{d\mathcal{H}_{n-m,m}}(N \boxtimes L(a^m), \Delta_{a^m}(M))$$

Proof. $N \boxtimes L(a^m)$ is in the block $(\ldots, a, \ldots, a)$ and so nonzero homomorphisms $N \boxtimes L(a^m) \to \operatorname{Res}_{n-m,m}^n M$ must land in the $(\ldots, a, \ldots, a)$ block of $\operatorname{Res}_{n-m,m}^n M$. But this is exactly $\Delta_{a^m}(M)$. ∎

Definition 1.5. *Given $a \in \Bbbk$ and $M \in d\mathcal{H}_n - mod$, let*

$$\epsilon_a(M) = \max\{m \geq 0 \,|\, \Delta_{a^m}(M) \neq 0\}$$

Proposition 1.6. *Let $m \geq 0$, $a \in \Bbbk$ and $N \in d\mathcal{H}_n - mod$ be <u>irreducible</u> with $\epsilon_a(N) = 0$ ($N_{\vec{b}} = 0$ if $b_n = a$). Set $M = \operatorname{Ind}_{n,m}^{n+m} N \boxtimes L(a^m)$. Then*

- *(i) $\Delta_{a^m}(M) \cong N \boxtimes L(a^m)$ (In particular $\operatorname{soc}(\Delta_{a^m}(M))$ is irreducible)*
- *(ii) $\Delta_{a^m}(\operatorname{hd}(M)) = \Delta_{a^m}(M)$ and $\operatorname{hd}(M)$(largest semisimple quotient) is irreducible.*
- *(iii) $\epsilon_a(\operatorname{hd}(M)) = m$ and all other composition factors L of M have $\epsilon_a(L) < m$.*

Proof. (i) From the unit of the adjunction from Lemma 1.4 we have a nonzero, injective (since N is simple) map

$$N \boxtimes L(a^m) \to \Delta_{a^m}(M)$$

Now using the shuffle lemma and the fact that when $\epsilon_a(N) = 0$ there is only one shuffle $\underline{b} \in \mathrm{wt}(N)$ and $(a, \dots, a)$ in which the last m spots are all a, we have that

$$\dim_{\Bbbk} N \boxtimes L(a^m) = \dim_{\Bbbk} \Delta_{a^m}(M)$$

and thus they are isomorphic.
(ii) Let $\mathrm{hd}(M) = M/I$. Because Δ_{a^m} is exact we have the SES

$$0 \to \Delta_{a^m}(I) \to \Delta_{a^m}(M) \to \Delta_{a^m}(\mathrm{hd}(M)) \to 0 \tag{1}$$

But since $\Delta_{a^m}(M) \cong N \boxtimes L(a^m)$ is simple it follows that $\Delta_{a^m}(I) = 0$. Moreover, any composition factor of $\Delta_{a^m}(\mathrm{hd}(M))$ will be a composition factor of $\Delta_{a^m}(M)$. From Lemma 1.4 we have that

$$\mathrm{Hom}_{n+m}(M, M/I) = \mathrm{Hom}_{n,m}(N \boxtimes L(a^m), \Delta_{a^m}(\mathrm{hd}(M))\,)$$

If $\mathrm{hd}(M)$ were not simple, then semisimplicity of M/I would give us at least 2 different maps on the LHS and thus if $N \boxtimes L(a^m)$ appears with mulitplicity 2 as a composition factor of $\Delta_{a^m}(\mathrm{hd}(M))$ and thus of $\Delta_{a^m}(M)$. But $\Delta_{a^m}(M) \cong N \boxtimes L(a^m)$ so this is impossible.

We have that $\Delta_{a^{m+1}}(M) = \Delta_{a^{m+1}}(\Delta_{a^m}(M)) = \Delta_{a^{m+1}}(N \boxtimes L(a^m)) = 0$ as $\epsilon_a(N) = 0$ and thus $\epsilon_a(\mathrm{hd}(M)) = m$. Eq. (1) shows $\Delta_{a^m}(I) = 0$ and thus $\epsilon_a(L) < m$ for all other composition factors L. ■

Lemma 1.7. *Let $M \in d\mathcal{H}_n - mod$ be irreducible, $a \in \Bbbk$. If $N \boxtimes L(a^m)$ is an irreducible submodule of $\Delta_{a^m}(M)$ for some $0 \le m \le \epsilon_a(M)$, then $\epsilon_a(N) = \epsilon_a(M) - m$.*

Lemma 1.8. *Let $M \in d\mathcal{H}_n{-}mod$ be irreducible, $a \in \Bbbk$ and $\epsilon := \epsilon_a(M)$. Then $\Delta_{a^\epsilon}(M)$ is isomorphic to $N \boxtimes L(a^\epsilon)$ for some irreducible $N \in d\mathcal{H}_{n-\epsilon}{-}mod$ with $\epsilon_a(N) = 0$.*

Proof. Choose any simple submodule $N \boxtimes L(a^\epsilon) \hookrightarrow \Delta_{a^\epsilon}(M)$. Then by Lemma 1.7 (with $m = \epsilon$) we have that $\epsilon_a(N) = 0$. By Lemma 1.4 we have a map

$$\mathrm{Ind}^n_{n-\epsilon,\epsilon} N \boxtimes L(a^\epsilon) \twoheadrightarrow M$$

which is surjective as M is irreducible by assumption. By exactness of Δ_{a^ϵ} we then have

$$\Delta_{a^\epsilon}(\mathrm{Ind}^n_{n-\epsilon,\epsilon} N \boxtimes L(a^\epsilon)) \twoheadrightarrow \Delta_{a^\epsilon}(M)$$

But by Proposition 1.6 (i), the LHS above is isomorphic to $N \boxtimes L(a^\epsilon)$ and thus the isomorphism as desired. ■

Theorem 1.9. *Let $M \in d\mathcal{H}_n{-}mod$ be irreducible, $a \in \Bbbk$. Then for any $0 \le m \le \epsilon_a(M)$, $\mathrm{soc}(\Delta_{a^m}(M))$ is an irreducible $d\mathcal{H}_{n-m,m}{-}mod$ of the form $L \boxtimes L(a^m)$ with $\epsilon_a(L) = \epsilon_a(M) - m$.*

Proof. When $m = \epsilon$ this is just the lemma above. Again let $\epsilon = \epsilon_a(M)$. Consider an irreducible summand

$$L \boxtimes L(a^m) \hookrightarrow \mathrm{soc}\,(\Delta_{a^m}(M)) \tag{2}$$

By Lemma 1.7 we have that $\epsilon_a(L) = \epsilon - m$. Thus taking the $x_{n-m}, \dots, x_{n-\epsilon+1}$ generalized $a-$eigenspace of both sides of Eq. (2) we obtain the inclusion

$$\Delta_{\epsilon-m}(L) \boxtimes L(a^m) \hookrightarrow \Delta_{a^\epsilon}(M)$$

Note that $\Delta_{\epsilon-m}(L)$ is simple by **??** and keeping track of the submodule structure the LHS is a $d\mathcal{H}_{n-m-(\epsilon-m),\epsilon-m,m}-$module and thus we have the inclusion of an irreducible

$$\Delta_{\epsilon-m}(L) \boxtimes L(a^m) \hookrightarrow \mathrm{Res}^{n-\epsilon,\epsilon}_{n-\epsilon,\epsilon-m,m}\Delta_{a^\epsilon}(M)$$

as submodules. But from Lemma 1.8 we have that $\Delta_{a^\epsilon}(M) = N \boxtimes L(a^\epsilon)$. We know that $\mathrm{soc}\left(\mathrm{Res}^{\epsilon}_{\epsilon-m,m}L(a^\epsilon)\right) = L(a^{\epsilon-m}) \boxtimes L(a^m)$ from the previous lecture and thus we have that

$$\mathrm{soc}\left(\mathrm{Res}^{n-\epsilon,\epsilon}_{n-\epsilon,\epsilon-m,m}\Delta_{a^\epsilon}(M)\right) = N \boxtimes L(a^{\epsilon-m}) \boxtimes L(a^m)$$

is simple and thus $\Delta_{\epsilon-m}(L)$ is unique and thus L is unique[1]. ■

2 Crystal Operators

Definition 2.1. *Let $M \in d\mathcal{H}_n-$mod be* <u>*irreducible*</u>*, define*

$$\widetilde{e_a}(M) = \mathrm{soc}(e_a(M)), \qquad \widetilde{f_a}(M) = \mathrm{hd}\left(\mathrm{Ind}^{n+1}_{n,1} M \boxtimes L(a)\right)$$

where $e_a(M) = \mathrm{Res}^{n-1,1}_{n-1} \circ \Delta_a(M)$.

Remark. In $d\mathcal{H}^{\Lambda_0}_n := d\mathcal{H}_n/(x_1) = \Bbbk[S_n]$ we have that $x_k \mapsto J_k$ where J_k is the $k-$th Jucys–Murphy element. Then e_a, "f_a" as defined above has a very nice decription when restricted to the Specht modules, e_a removes a box of content a while "f_a" adds a box of content a.

Remark. "f_a" is in quotations above because it's not defined.

Lemma 2.2. $\widetilde{e_a} : d\mathcal{H}_n-irr \to d\mathcal{H}_{n-1}-irr$ *and* $\widetilde{f_a} : d\mathcal{H}_n-irr \to d\mathcal{H}_{n+1}-irr$

Proof. We just show the case $\widetilde{e_a}$. Let $L \hookrightarrow e_a(M)$ be an irreducible submodule. We need to show L is unique. First note that as a set, $e_a(M) = \Delta_a(M) \subset M$. We claim that L is in fact a $d\mathcal{H}_{n-1,1}$ submodule, aka stable under the action of x_n. Note

(1) $z = x_1 + \ldots + x_n$ is central in $d\mathcal{H}_n$ it acts by a scalar on the irreducible $d\mathcal{H}_n-$module M and thus on any subset L.

(2) $z' = x_1 + \ldots + x_n$ is central in $d\mathcal{H}_{n-1}$ it acts by a scalar on the irreducible $d\mathcal{H}_{n-1}-$module L.

(3) Therefore $x_n = z - z'$ acts by a scalar on L.

(4) $L \subset \Delta_a(M)$ as a set, so L is a subset of the generalized $a-$eigenspace for x_n and since x_n acts by a scalar that scalar must be a.

(5) Therefore as a $d\mathcal{H}_{n-1,1}$ module $L = L \boxtimes L(a) \subset \Delta_a(M)$. This is irreducible and thus contributes to the socle and by Theorem 1.9 the socle is irreducible so L is unique.

■

Proposition 2.3. *Let $M \in d\mathcal{H}_n-$mod be* <u>*irreducible*</u>*, $a \in \Bbbk$. Then*

(a) $\mathrm{soc}\,(\Delta_{a^m} M) \cong (\widetilde{e_a}^m(M)) \boxtimes L(a^m)$.

[1] The functors Δ_{a^k}, Res are all restriction functors so the inclusion of another $L \boxtimes L(a^m)$ would genuinely produce a different factor.

(b) $\text{hd}\left(\text{Ind}_{n,m}^{n+m} M \boxtimes L(a^m)\right) \cong \widetilde{f_a}^m(M)$.

Proof. (*a*) If $m > \epsilon_a(M)$ then both parts in the equality are 0. So let $m \le \epsilon_a(M)$ [TODO] ■

Lemma 2.4 (Crystal)**.** *Let $A \in d\mathcal{H}_n$−mod and $B \in d\mathcal{H}_{n+1}$ be irreducible modules and $a \in \Bbbk$. Then $\widetilde{f_a}(A) = B \iff \widetilde{e_a}(B) = A$.*

Corollary 2.5. *Let $M, N \in d\mathcal{H}_n$−mod be irreducible. Then $\widetilde{e_a}(M) \cong \widetilde{e_a}(N) \iff M \cong N$ assuming $\epsilon_a(M), \epsilon_a(N) > 0$.*

Proof. $\implies$ Suppose $\widetilde{e_a}(M) \cong \widetilde{e_a}(N)$. By Lemma 2.4 with $B = M, A = \widetilde{e_a}(N)$ we have that $\widetilde{f_a}\left(\widetilde{e_a}(N)\right) = M$. But we can apply Lemma 2.4 again with $B = N, A = \widetilde{e_a}(M)$ to obtain $\widetilde{f_a}\left(\widetilde{e_a}(M)\right) = N$ and thus $M \cong N$ as desired. ■

Theorem 1 (Vazirani)
The map $\text{ch} : K_0(d\mathcal{H}_n - \text{mod}) \to K_0(P_n - \text{mod})$ *is injective where* $\text{ch}(M) = [\text{Res}_{P_n}^n M]$.

Proof. It suffices to show that $\{\text{ch}(L)\}_{L \text{ irr}}$ is L.I. over $\mathbb{Z}$. Proceed by induction on n. Suppose we have

$$\sum_L c_L \,\text{ch}(L) = 0 \qquad c_L \in \mathbb{Z} \tag{3}$$

for some simple $L \in d\mathcal{H}_n$−mod. Choose $a \in \Bbbk$, we will show by downward induction that $c_L = 0$ if $\epsilon_a(L) = k$ where $k = n, \ldots, 1$. Doing this for all a will then complete the proof. Because Δ_{a^n} is exact, it descends to a map $K_0(P_n - \text{mod}) \to K_0(P_n - \text{mod})$ and commutes with Res. The only simple in the block $(a, \ldots, a)$[2] is $L(a^n)$ and thus applying Δ_{a^n} to Eq. (3), we see that the coefficient of $\text{ch}L(a^n)$ is zero completing the base case $k = n$.

Now suppose that $c_L = 0$ for all L with $\epsilon_a(L) > k$, applying Δ_{a^k} to Eq. (3) we have

$$\sum_{L \text{ s.t. } \epsilon_a(L)=k} c_L \,\text{ch}(\Delta_{a^k}(L)) = 0 \tag{4}$$

because $c_L = 0$ if $\epsilon_a(L) > k$ by induction and $\Delta_{a^k}(L) = 0$ if $\epsilon_a(L) < k$. Since $\epsilon_a(L) = k$ Lemma 1.8 tells us that $\Delta_{a^k}(L)$ is simple and thus equal to it's socle. From Proposition 2.3 we then see that

$$\Delta_{a^k}(L) \cong \left(\widetilde{e_a}^k(L)\right) \boxtimes L(a^k)$$

and thus we can factor out a $[L(a^k)]$ from Eq. (4) to obtain

$$\sum_{L \text{ s.t. } \epsilon_a(L)=k} c_L \,\text{ch}(\widetilde{e_a}^k(L)) = 0$$

We know that $\widetilde{e_a}^k(L) \in d\mathcal{H}_{n-k}$−irr so by induction all the $c_L = 0$ assuming that $\left\{\widetilde{e_a}^k(L)\right\}$ are all distinct. But this is exactly what Corollary 2.5 tells us so we are done. ■

[2] n times

3 Misc Results

Proposition 3.1. *Let $M \in d\mathcal{H}_n - mod$ be <u>irreducible</u>, then* $\text{soc}\left(\text{Res}^n_{n-1} M\right)$ *is multiplicity-free.*

Proof. We have that $\text{Res}^n_{n-1} M = \bigoplus_{a \in \Bbbk} e_a(M)$ with all but finitely many summands zero and thus

$$\text{soc}\left(\text{Res}^n_{n-1} M\right) = \bigoplus_{a \in \Bbbk} \text{soc}(e_a(M)) = \bigoplus_{a \in \Bbbk} \widetilde{e_a}(M)$$

where we have used tha soc commutes with direct sum (see [Modular Representation Theory of Finite Groups, Exercise 24.5] by Lassueur, Farrell). Alternatively in this case each irreducible is contained in a unique block so must be contained inside $\text{soc}(e_a(M))$ for some a and thus soc commutes with the direct sum above.

Now we know $\widetilde{e_a}(M)$ is irreducible and for different $a \in \Bbbk$, $\widetilde{e_a}(M)$ are in different blocks and thus can't be isomorphic to each other and thus $\text{soc}\left(\text{Res}^n_{n-1} M\right)$ is multiplicity free as desired. ■

4 Categorification of $U(\widehat{\mathfrak{sl}_p})$

Definition 4.1. *Given $\Bbbk$ let $I := \mathbb{Z} \cdot I \subset \Bbbk$. As a set $I = \mathbb{Z}/p\mathbb{Z}$ where $p = \text{char}\, \Bbbk$.*

Definition 4.2. *$M \in d\mathcal{H}_n -mod$ is called integral if all the eigenvalues of $\{x_i\}_{i=1}^n$ are in I. Let $d\mathcal{H}_n - \text{mod}_I$ be the full subcategory of $d\mathcal{H}_n -mod$ consisting of all integral modules.*

Theorem 2

Let $K_0(d\mathcal{H}_\Bbbk) = \bigoplus_{n \geq 0} K_0(d\mathcal{H}_n/_\Bbbk - \text{mod}_I)$ and let $K_\oplus(d\mathcal{H}_\Bbbk) = \bigoplus_{n \geq 0} K_\oplus(d\mathcal{H}_n/_\Bbbk - \text{pmod}_I)$. Then there are isomorphisms of Hopf algebras

$$U_\mathbb{Z}(\widehat{\mathfrak{sl}_p}^+) \xrightarrow{\sim} K_\oplus(d\mathcal{H}_\Bbbk) \qquad U^*_\mathbb{Z}(\widehat{\mathfrak{sl}_p}^+) \xrightarrow{\sim} K_0(d\mathcal{H}_\Bbbk)$$

where $p = \text{char}\, \Bbbk$, s.t.

$$\text{CB}_{\widehat{\mathfrak{sl}_p}} \longleftrightarrow \{[P]_{P \text{ indec}}\} \qquad \text{DCB}_{\widehat{\mathfrak{sl}_p}} \longleftrightarrow \{[L]_{L \text{ irr}}\}$$

and $\widetilde{E_a}, \widetilde{F_a} \longleftrightarrow \widetilde{e_a}, \widetilde{f_a}$.